Moist Convective Self-Similarity: Scaling Dynamics of Atmospheric Turbulence in Cumulonimbus Clouds

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Abstract: The Scaling behavior of atmospheric turbulence is profoundly influenced by moist convection, where latent heat release and moisture transport drive complex multi-scale interactions. While classical turbulence theory assumes dry, incompressible flow with stationary scaling exponents, the intense thermodynamic and micro physical processes within cumulonimbus clouds challenged this paradigm. This study investigates how latent heat release and moisture content modify the scaling laws of velocity and temperature structure functions in moist convective turbulence.

1.Introduction:

We are all familiar with the name **"CUMULONIMBUS"** which indicates a special type of cloud. This type of clouds are towering vertical, often associated with thunderstorms, heavy rain, lightning, hail, extending from near the surface to the tropopause (10- 16 km), and sometimes it forms tornadoes in the specific region with specific conditions. This type of clouds are considered as a natural laboratory of turbulence. Unlike classical turbulence, moist convection adds latent heat release, moisture transport, and buoyancy- driven forcing at multiple scales. These processes suggest a modification in self-similar scaling laws, making turbulence exponents time-dependent.

2.Moist Convective Turbulence in Cumulonimbus Clouds:

2.1 Formation:

Scale separation: Energy is injected at large scales by external forcing (e.g., solar heating), cascades through mesoscale (cloud/cluster scale, ~10–50 km), and organizes at smaller scales by microphysical processes (1–1000 m).

Updrafts in cumulonimbus clouds organize at preferred scales, sometimes determined by boundary layer depth, CAPE (convective available potential energy), and local humidity variations.

Statistical intermittency: Probability density functions of vertical velocity and temperature often show long tails/skewness, especially during storm formation**.**

2.2 Physical Concepts of Multi-Scale Turbulence:

Characteristics: Multi-scale turbulence in cumulonimbus involves a broad spectrum of eddy sizes, from sub-millimetre droplets to kilometre-scale convective towers. The smallest eddies give a cauliflower appearance to the cloud, and the mixing is driven by small-scale processes and large-scale vertical motions.

Microphysics: Water vapor condenses on cloud nuclei, releasing energy and triggering further ascent. Ice formation occurs at higher altitudes, affecting the buoyancy and other thermodynamic processes in upper levels.

Scaling Behaviour: The sales-similarity of turbulent structure changes due to phase transitions (liquid / ice) and energy input form condensation and freezing, leading to a highly random distribution of velocity and temperature fluctuations.

Latent Heat Release: Latent heat release drives buoyancy, amplifies updrafts, and feeds turbulent cascades.

Moisture Transport: Moisture transport modifies local thermodynamic gradients; influences both turbulence generation and decay.

Radiative Cooling/Heating: Alters stratification, radiative cooling / heating promotes new turbulence, regulates cloud life cycles.

2.3 Statistical Implications:

Turbulence within cumulonimbus clouds is non-stationary, strongly intermittent, and scale-coupled.

Scaling laws (energy dissipation, structure functions) become time-variable and clustering emerges (mesoscale convective systems).

Spontaneous symmetry breaking occurs on faster timescales compared to radiative–convective equilibrium settings.

3. Classical Turbulence Vs. Moist Convective Turbulence:

3.1 Classical (For Dry and Incompressible) Turbulence:

Governed largely by Kolmogorov's theory, (Kolmogorov,1941) classical turbulence assumes stationarity and self-similarity across scales, with energy cascading from large to smaller eddies and scaling exponents derived from inertial-range statistics**.**

In Dry Air, Navier–Stokes equation is applied for atmospheric turbulence (incompressible form with Coriolis force and buoyancy terms, typical in meteorology):

$$\frac{\partial u}{\partial t} + (u.\nabla)u + fk \times u = -\frac{1}{\rho}\nabla p + \nu\nabla^2 u + B _____ (1)$$

where: $u = (u, v, w) \rightarrow velocity\ vector\ of\ air\ (x, y, z\ components)$

$$\frac{\partial u}{\partial t} \rightarrow local\ acceleration\ (\ time\ rate\ of\ change\ )$$

$$(u.\nabla)u \rightarrow advection\ (\ transport\ of\ momentum\ by\ flow\ itself)$$

$$fk \times u \rightarrow coriolis\ forces\ (\ rotation\ of\ Earth\ , important\ in\ atmospheric\ flows)$$

$$f = coriolis\ parameter$$

$$k = vertical\ unit\ vector$$

$$-\frac{1}{\rho}\nabla p \rightarrow pressure\ gradient\ force$$

$\nu\nabla^2 u \rightarrow$ viscous diffusion of momentum (molecular viscosity, but in turbulence we often replace this with an eddy viscosity).

B→Buoyancy force (arises from density or temperature fluctuations, usually modeled with Boussisesq approximation as $B = g\frac{\theta'}{\theta_0}\ k$

Structure functions of velocity, $S_p(R) =< |\partial_u(r)|^p >$ follow statistical scaling, often with exponents $\zeta_p = \frac{p}{3}$ in the inertial subrange.

3.2 Moist Convective Turbulence:

Moist turbulence in clouds introduces phase changes(condensation/evaporation), nonlinearity due to latent heat release, and active microphysics.

The scalar fields (temperature, humidity) are more dynamically coupled.

Scaling exponents and energy transfer pathways are modified by these feedbacks**.**

4. Modelling Scaling Behaviour:

4.1 General Form of the Continuity Equation

For any fluid, the continuity equation fundamentally states:

$$\frac{\partial \rho}{\partial t} + \nabla.(\rho v) = 0 _____ (2)$$

where:

ρ is the fluid density

v is the velocity field.

This equation ensures that, as air and moisture move and change phases in convective clouds the total mass is conserved.

4.2 Boussinesq Approximation (Atmospheric Convection)

In atmospheric turbulence, especially when using the Boussinesq approximation (applicable for small density variations, typical in deep convection), the continuity equation simplifies to:

$$\nabla . V = 0 \quad \text{(3)}$$

Here, incompressibility implies that the divergence of velocity is zero i.e. whatever air flows into a region must flow out, so local density changes are negligible except where buoyancy is concerned (e.g., due to latent heat release).

4.3 Moisture Continuity Equation

For a humid atmosphere, one must also account for specific humidity q or mixing ratio, and its transport:

$$\frac{\partial q}{\partial t} + \nabla . (qv) = S_q \quad \text{(4)}$$

S_q encompasses sources/sinks like condensation (which converts water vapor to liquid, releasing latent heat) and evaporation. This equation is crucial inside cumulonimbus clouds, where intense phase changes occur.

4.4 Governing Equations

The basic model for moist convection builds from the air's equations of motion, continuity, and thermodynamic state, but includes extra terms for (specific) humidity q and latent heat L_v :

$$\frac{\partial t}{\partial u} + (u . \nabla)u = -\frac{1}{\rho_0} \nabla p + g \frac{\theta_v - \bar{\theta}_v}{\bar{\theta}_v} + \nu \nabla^2 u \quad \text{(5)}$$

$$\frac{\partial T}{\partial t} + (u . \nabla)T = Q_{rad} + \frac{L_v \partial q}{C_p \partial t} \quad \text{(6)}$$

$$\frac{\partial T}{\partial t} + (u . \nabla)q = S_q \quad \text{(7)}$$

where:

- θ_v : virtual potential temperature (accounts for humidity)
- Q_{rad} : radiative heating/cooling

S_q: source/sink term for moisture (condensation/evaporation).

4.5 Scaling Structure Functions:

In moist turbulence, the second-order structure function for velocity or temperature is altered:

$$S_2^u(r) = < [u(x+r) - u(x)]^2 > _____ (8)$$

$$S_2^T(r) = < [T(x+r) - T(x)]^2 > _____ (9)$$

The scaling exponents, ζ_2^u and ζ_2^T, are not stationary—they evolve due to intermittency caused by heating and microphysical processes (where classical turbulence assumes they are constants).

4.6 Scaling Laws:

Boussinesq and Non-Boussinesq Approximation: The development of strong thermals is modeled by equations for mass, momentum, and buoyancy conservation. In the weak thermal and Boussinesq case, traditional Batchelor, Morton, and Turner scaling are recovered.

4.6.1 Velocity Structure Function: Classical turbulence theory estimates the second-order velocity structure function as:

$$S_2(r) = C_2(\epsilon r)^{\frac{2}{3}} _____ (10)$$

with ϵ as the turbulent energy dissipation rate and r as the separation scale. In moist convection, latent heat release and phase changes introduce nonstationary scaling exponents and intermittency.

4.6.2 Temperature Structure Function: For temperature variations,

$$S_\theta(r) = C_\theta(\chi^r)^2 _____ (11)$$

where χ is the temperature dissipation rate, and the exponent n may deviate from classical Kolmogorov scaling due to non-linear moisture-energy coupling.

4.6.3 Latent Heat Impact: Moist convection features additional energy input due to latent heat release at condensation: Q = mL, where m is mass and L is latent heat per unit mass.

4.6.4 Self-Similarity: In classical theory, self-similarity implies that eddy statistics at different scales have similar distribution shapes when rescaled.

In moist convection, self-similarity arises but the scaling exponents reflect the effects of condensation, latent heat, and moist instability—often yielding multifractal behaviour rather than simple mono-fractal scaling.

4.6.5 Empirical Findings: Observations and simulations show persistent vertical velocity structures and organized motions (e.g., updrafts, downdrafts), especially in intense cumulonimbus clouds.

Structure function scaling in moist convection often deviates from Kolmogorov, showing steeper slopes and intermittent bursts linked to cloud microphysics, analogous to multifractal cascades.

Evaporation, precipitation, and radiative effects further modify the self-similar structure within the cloud.

10.Conclusion:

Moist convection rewrites scaling laws of atmospheric turbulence in cumulonimbus clouds, with scaling exponents varying due to latent heat, moisture, and radiative feedbacks Understanding these dynamics is crucial for weather forecasting, climate modelling, and assessing extreme rainfall events, making continued study of moist convective scaling an essential frontier in atmospheric science.

In future mathematical/ as well as statistical tools (structure functions, PDFs, multifractal analysis, spectral methods) may be introduced to reveal strong intermittency, multifractality, and non-stationarity.

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